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FAMILIES OF FINITE SETS WITH THREE INTERSECTIONS

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Let X be a finite set of n elements and $\mathcal{F}$ a family of k-subsets of X. Suppose that for a given set L of non-negative integers all the pairwise intersections of members of $\mathcal{F}$ have cardinality belonging to L. Let $m(n, k, L)$ denote the maximum possible cardinality of $\mathcal{F}$. This function was investigated by many authors, but to determine its exact value or even its correct order of magnitude appears to be hopeless. In this paper we investigate the case $|L|=3$. We give necessary and sufficient conditions for $m(n,k,L)=O(n)$ and $m(n,k,L)\geqq O(n^2)$, and show that in some cases $m(n,k,L)=O(n^{3/2})$, which is quite surprising.

1. Introduction

Let $0\leqq l_1<l_2<\ldots<l_s<k<n$ be integers, and X a finite set of cardinality n. We say that the family $\mathcal{F}$ of subsets of X is an $(n, k, \{l_1, \ldots, l_s\})$-system if for every $F_1, F_2\in\mathcal{F}$ we have $|F_1|=k$, $|F_1\cap F_2|\in\{l_1, \ldots, l_s\}$.

Let us denote by $m(n, k, \{l_1, \ldots, l_s\})$ the maximum cardinality of an $(n, k, \{l_1, \ldots, l_s\})$-system.

Theorem 1. (Ray—Chaudhuri, Wilson [7])

$$m(n, k, \{l_1, \ldots, l_s\}) \leqq \binom{n}{s}. \quad \blacksquare$$

Theorem 2. (Deza, Erdős, Singhi [3])

$$m(n, k, \{0, a\}) \leqq \frac{n}{k}\frac{n-a}{k-a},$$

moreover, if a is not a divisor of k then

$$m(n, k, \{0, a\}) \leqq n. \quad \blacksquare$$

For large n these theorems were generalized by

Theorem 3. (Deza, Erdős, Frankl [2]). *Suppose* $n>n_0(k)$ *then*

$$m(n, k, \{l_1, \ldots, l_s\}) \leqq \prod_{1\leqq i\leqq s} \frac{n-l_i}{k-l_i},$$

AMS subject classification (1980): 05 A 05, 05 C 35, 05 C 65 *moreover if* $(l_2-l_1)|(l_3-l_2)|\dots|(l_s-l_1)|(k-l_s)$ *does not hold then for some constant* $c(k)$ *we have*

$$m(n, k, \{l_1, \dots, l_s\}) \leqq c(k)n^{s-1}. \quad \blacksquare$$

The next result generalizes the second part of Theorem 2.

Theorem 4. (Babai, Frankl [1].) *If* g.c.d. $\{l_1, \dots, l_s\}\nmid k$, *then*

$$m(n, k, \{l_1, \dots, l_s\}) \leqq n.$$

However if there exist non-negative integers $d_1, \dots, d_s$ *such that* $k = d_1 l_1 + d_2 l_2 + \dots + d_s l_s$ *then*

$$m(n, k, \{l_1, \dots, l_s\}) \geqq \frac{n^2}{4k^2}. \quad \blacksquare$$

For some more results on $(n, k, \{l_1, \dots, l_s\})$-systems cf. [5].

2. The statement of the results

We write $f(n) = O(g(n))$ if there exist positive constants c_1, c_2 such that $f(n) \geqq c_1 g(n)$, and $g(n) \leqq c_2 f(n)$ hold for $n > n_0$. In [6] it is proved that

$$m(n, k, \{l_1, \dots, l_s\}) = O(m(n, k-l_1, \{0, l_2-l_1, \dots, l_s-l_1\})).$$

Therefore from now on we always assume $l_1 = 0$. Trivially $m(n, k, \{0\}) = \left\lfloor \frac{n}{k} \right\rfloor = O(n)$.

For $s=2$ from Theorems 2 and 4 we deduce

$$m(n, k, \{0, a\}) = \begin{cases} O(n^2) & \text{if } a|k \\ O(n) & \text{if } a\nmid k. \end{cases}$$

In this paper we are concerned with the case $s=3$. We shall see that the situation gets much more complicated. The following theorem gives a necessary and sufficient condition for $m(n, k, \{0, a, b\}) = O(n)$ and $m(n, k, \{0, a, b\}) \geqq O(n^2)$. As we shall see, in a lot of cases $m(n, k, \{0, a, b\}) = O(n^{3/2})$ holds.

To state the main result we need some definitions. Set $b_0 = \lfloor b/a \rfloor$, and $b_1 = b - b_0 a$ i.e. $b = b_0 a + b_1$ with $0 \leqq b_1 < a$. We say a, b, k satisfy $(*)$ if there exist non-negative integers f, g with $k = fa + gb_1$ and $m(f, b_0, \{0, 1\}) \geqq g$.

If a, b, k satisfy the condition $(*)$ we assume always that f is chosen to be maximal. Note that in the case $g=0$ we have $a|k$, and in the case $g=1$ we have $k = b + (f-b_0)a$. Thus in view of Theorem 4 in these cases $m(n, k, \{0, a, b\}) \geqq O(n^2)$ holds.

Theorem 5.

(i) *If* a, b, k *do not satisfy* $(*)$, *then*

$$m(n, k, \{0, a, b\}) = O(n).$$

(ii) *If* a, b, k *satisfy* $(*)$, $g \geqq 3$, $f < b_0 g$ *and* $b-a$ *does not divide* $k-a$, *then*

$$m(n, k, \{0, a, b\}) \leqq O(n^{3/2}),$$

$$m(n, k, \{0, a, b\}) \geqq O(n^{g/(g-1)}).$$

(iii) *in all the remaining cases we have*

$$m(n, k, \{0, a, b\}) \geqq O(n^2).$$

Remark 1. The case (iii) of the theorem follows from the second part of Theorem 4. In fact, either $b-a$ divides $k-a$, and the statement follows from $m(n, k, \{0, a, b\}) \geqq m(n-a, k-a, \{0, b-a\})$, or we have $k = fa + gb_1 = (f - gb_0)a + gb$, and we can apply Theorem 4 as long as $f \geqq gb_0$. The only missing cases would be $g = 1$ and $g = 2$. However $(*)$ implies $f \geqq b_0$ if $g = 1$, and $f \geqq 2b_0 - 1$ if $g = 2$. Thus only the choice $f = 2b_0 - 1$, $g = 2$ would violate $f \geqq gb_0$. However, in this latter case $k = 2b - a$ i.e. $b-a$ divides $k-a$, a case which we have already considered.

Remark 2. The second part of (ii) will be proved by construction, we will give another construction giving $O(n^{3/2})$ sets for a lot of the cases.

Remark 3. In view of Theorem 1 we always have $m(n, k, \{0, a, b\}) \leqq \binom{n}{3}$, Theorem 2 gives $m(n, k, \{0, a, b\}) \leqq O(n^2)$ unless $a|(b-a)|(k-a)$. However, to give a necessary and sufficient condition seems to be very hard even for $m(n, k, \{0, 1, 3\}) = O(n^2)$.

For the proofs we need the following:

Theorem 6. (Erdős, Rado [4].) *Suppose* $\mathscr{A}$ *is a collection of distinct* k*-element sets,* $k, d \geqq 2$, *are integers. If* $|\mathscr{A}| > d^k k!$ *then* $\mathscr{A}$ *contains a* Δ*-system of cardinality* $d+1$, *i.e., sets* $A_1, \ldots, A_{d+1}$ *such that for some set* K, *called the kernel, we have* $A_i \cap A_j = K$ *for every* $1 \leqq i < j \leqq d+1$. ∎

The next lemma was proved in [2].

Lemma 1. *Suppose both* K_1 *and* K_2 *are kernels of* Δ*-systems consisting of* $k+1$ *members of an* $(n, k, \{l_1, \ldots, l_s\})$*-system* $\mathscr{F}$, *each. Then* $|K_1 \cap K_2| \in \{l_1, \ldots, l_s\}$, *and also for every* $F \in \mathscr{F}$ *we have* $|F \cap K_1| \in \{l_1, \ldots, l_s\}$. ∎

3. Proof of (i) and of the first part of (ii)

Let $\mathscr{F}$ be an $(n, k, \{0, a, b\})$-system of maximum cardinality. Let us define $\mathscr{A}$ ($\mathscr{B}$) as the family of those a-subsets (b-subsets) which are kernels of Δ-systems of cardinality $k+1$ (k^2+1) consisting of members of $\mathscr{F}$, i.e., if $B \in \mathscr{B}$, then there exist $F_0, \ldots, F_{k^2} \in \mathscr{F}$ such that $F_i \cap F_j = B$ for $0 \leqq i < j \leqq k^2$. Let us set also $\mathscr{K} = \mathscr{A} \cup \mathscr{B}$. Let us define $\mathscr{F}_0 = \{F \in \mathscr{F}\colon F \neq \bigcup_{\substack{K \subset F \\ K \in \mathscr{K}}} K\}$, that is $\mathscr{F}_0$ is the collection of those members of $\mathscr{F}$ which are not obtained as the union of kernels of large Δ-systems.

Proposition 1.

$$|\mathscr{F}_0| \leqq k! k^{2k} n.$$

Proof. We associate with every $F\in\mathcal{F}_0$ an element $x(F)\in X$ which is not contained in $\bigcup_{K\subset F, K\in\mathcal{K}} K$. It is sufficient to show that no $x\in X$ is associated with more than $k!\,k^{2k}$ sets $F\in\mathcal{F}_0$. Let $\mathcal{F}(x)$ denote the collection of sets F, associated with x, that is $x=x(F)$. In view of Theorem 6 it is enough to show that $\mathcal{F}(x)$ contains no Δ-systems of cardinality k^2+1. Suppose the contrary and let $F_1, \dots, F_{k^2+1}$ be the members of such a Δ-system. As $x\in F_i$ for $i=1, \dots, k^2+1$, the kernel K of the Δ-system also contains x. By definition, $K\in\mathcal{K}$ which contradicts the choice of $x=x(F_1)$. ∎

Set now $\mathcal{F}_1=\mathcal{F}-\mathcal{F}_0$. By Lemma 1 we know that if $B_1, B_2\in\mathcal{B}$, then $|B_1\cap B_2|\in\{0, a\}$. We say that B_1, B_2 is a bad pair if $|B_1\cap B_2|=a$, but $(B_1\cap B_2)\notin\mathcal{A}$.

Proposition 2. *The number of bad pairs is at most* $(b!\,k^b)^2n$.

Proof. For an $x\in X$ let us define $\mathcal{B}(x)=\{B\in\mathcal{B}\colon x\in B\}$. Suppose that $\mathcal{B}(x)$ contains a Δ-system of cardinality $k+1$, and let $B_1, \dots, B_{k+1}$ form such a Δ-system with kernel A. By Lemma 1 we have $|A|=a$. We want to show $A\in\mathcal{A}$. In order to do that we have to find a Δ-system of cardinality $k+1$ with kernel A, among the members of $\mathcal{F}$. Suppose we have chosen $F_1, \dots, F_t$ such that $B_i\subset F_i\in\mathcal{F}$ for $i=1, \dots, t$, and the F_i have pairwise intersection A. Since $B_{t+1}\in\mathcal{B}$, there exist $F^0, \dots, F^{k^2}$ satisfying $B_{t+1}\subset F^i\in\mathcal{F}$, and the sets F^i-B_{t+1} are pairwise disjoint for $i=0, \dots, k^2$. Now setting $Y_t=F_1\cup\dots\cup F_t$, we have $|Y_t|<kt\leqq k^2$ for $t\leqq k$. Thus Y_t cannot intersect all the k^2+1 pairwise disjoint sets F^i-B_{t+1} e.g. $(F^0-B_{t+1})\cap Y_t=\emptyset$. Set $F_{t+1}=F^0$, and continue. At last we obtain the desired Δ-system $F_1, \dots, F_{k+1}$ with kernel A, proving $A\in\mathcal{A}$.

Suppose now $B\in\mathcal{B}(x)$. Then in view of Lemma 1 we have $|A\cap B|=a$, and consequently $A\subset B$. Thus all the members of $\mathcal{B}(x)$ contain A. Applying again Lemma 1 we deduce that the intersection of any two members of $\mathcal{B}(x)$ is just A. Thus $\mathcal{B}(x)$ contains no bad pair.

We have proved that if $\mathcal{B}(x)$ contains a bad pair then it contains no Δ-system of cardinality $k+1$. Thus by Theorem 6 for such $x\in X$ we have $|\mathcal{B}(x)|\leqq b!\,k^b$.

Associating with every bad pair (B, B') an element $x\in(B\cap B')$ we infer that the number of bad pairs is at most $\sum_x |\mathcal{B}(x)|^2$, where the summation runs over all $x\in X$ which are associated with some bad pair. As for such x we have showed $|\mathcal{B}(x)|\leqq b!\,k^b$, the statement of the proposition follows. ∎

Now we give the proof of (i). First observe that every bad pair (B, B') is contained in at most one member of $\mathcal{F}$ (otherwise two such sets would have intersection of cardinality at least $2b-a>b$). Define $\mathcal{F}_2$ as the collection of those members of $\mathcal{F}_1$ which contain no bad pair. Then in view of Propositions 1 and 2 we have

$$(1)\qquad |\mathcal{F}_2| \geqq |\mathcal{F}|-(k!\,k^{2k}+(b!\,k^b)^2)n.$$

Hence we may assume $\mathcal{F}_2$ is non-empty, let F be a member of it. Let us set $\mathcal{A}(F)=\{A\in\mathcal{A}\colon A\subset F\}$. Then by Lemma 1 the members of $\mathcal{A}(F)$ are pairwise disjoint: let $A_1, \dots, A_q$ be these sets and D their union. Let us choose further a minimal collection, say $B_1, \dots, B_r$ of members of $\mathcal{B}$ such that $(F-D)\subseteqq(B_1\cup\dots\cup B_r)\subseteqq F$. It is possible because $F\in\mathcal{F}_2\subseteqq\mathcal{F}_1$.

As $F\in\mathcal{F}_2$, all the pairs (B_i, B_j) are good for $1\leqq i<j\leqq r$. Thus $B_i\cap B_j$ is either empty or one of the A_t's for $1\leqq t\leqq q$. Hence the sets $B_1-D, \dots, B_r-D$ partition $F-D$, moreover $|B_i-D|=b-c_ia$ for some integer c_i, $0\leqq c_i\leqq b_0=\lfloor b/a\rfloor$. For every

$B_i - D$ choose an arbitrary partition $E_0^i, \ldots, E_{b_0-c_i}^i$ of it with $|E_0^i| = b_1$, and all the other sets of cardinality a. Now F is partitioned into sets of cardinality a and b_0, which proves the first part of $(*)$—actually $g = r$, $f = q + \sum_{1 \leqq t \leqq r} (b_0 - c_i)$.

To prove the second part of $(*)$ set $\mathcal{E} = \{A_1, \ldots, A_q, E_1^1, \ldots, E_{b_0-c_1}^1, E_1^2, \ldots$ $\ldots, E_{b_0-c_r}^r\}$. Then $\mathcal{E}$ has f elements. Now define for every j, $1 \leqq j \leqq g$ the set $\mathcal{E}_j = \{E \in \mathcal{E}\colon E \subseteqq B_j\}$. Then $\mathcal{E}_j$ has cardinality b_0, and for $1 \leqq j \neq j' \leqq g$ we have $|\mathcal{E}_j \cap \mathcal{E}_{j'}| \in \{0, 1\}$, proving $m(f, b_0, \{0, 1\}) \geqq g$, concluding the proof of $(*)$.

Now we turn to the first part of (ii). In view of (1) it will be sufficient to show $|\mathcal{F}_2| \leqq O(n^{3/2})$. Maintaining the previous notations and using the condition $f < b_0 g$, we deduce that for an arbitrary $F \in \mathcal{F}_2$ we can find B_j, B_{j^*} such that $1 \leqq j \neq j^* \leqq g = r$ and $B_j \cap B_{j^*} = A_t$ for some $1 \leqq t \leqq q$. Moreover, the members of $\mathcal{F}$ intersect in at most b elements, whence every such pair (B_j, B_{j^*}) is contained in at most one member of $\mathcal{F}$. Thus it will be sufficient to prove:

Proposition 3. *The number of pairs* (B, B') *with* $B, B' \in \mathcal{B}$, $\exists F \in \mathcal{F}$, $(B \cup B') \subseteqq F$, $(B \cap B') \in \mathcal{A}$ *is at most* $O(n^{3/2})$.

Proof. Let us construct a graph $\mathcal{H}$ on the vertex-set $\mathcal{A} \cup \mathcal{B}$, and with the following edges: (A, B) if $A \subset B$; (B, B') if $(B \cap B') \in \mathcal{A}$, and $B \cup B'$ is contained in some $F \in \mathcal{F}$. Actually we are interested in the number of these latter edges, however this is the same as the number of triangles which have one vertex in $\mathcal{A}$. Let $d(A)$ denote the degree of A, i.e., $d(A) = |\{B \in \mathcal{B}\colon A \subset B\}|$. Let further $t(A)$ denote the number of triangles in $\mathcal{H}$ through the vertex A. Setting $\mathcal{F}(-A) = \{F - A\colon A \subset F \in \mathcal{F}\}$, we have $t(A) \leqq |\mathcal{F}(-A)|$. As $\mathcal{F}(-A)$ is an $(n-a, k-a, \{0, b-a\})$-system and $b-a$ does not divide $k-a$, by Theorem 2 we have $|\mathcal{F}(-A)| \leqq n$. We conclude

$$t(A) \leqq \min\left\{n, \binom{d(A)}{2}\right\}. \tag{2}$$

As $\mathcal{A}$ is an $(n, a, \{0\})$-system, we have $|\mathcal{A}| \leqq n/a$. Similarly $a \nmid b$, Lemma 1 and Theorem 2 yield $|\mathcal{B}| \leqq n$.

Let $a(i)$ $(A(i))$ denote the number of vertices in $\mathcal{A}$ which have in $\mathcal{H}$ degree i (at least i), respectively. We claim:

$$A(i) \leqq b_0 n/i. \tag{3}$$

In fact the contrary would imply that there exists a vertex $B \in \mathcal{B}$ which is connected to at least $b_0 + 1$ vertices, say $A_1, \ldots, A_{b_0+1}$ in $\mathcal{A}$, i.e., $B \supseteqq (A_1, \ldots, A_{b_0+1})$. However, this latter union has cardinality $(b_0+1)a > b$, and this contradiction proves (3).

Let us denote $\min\left\{n, \binom{d(A)}{2}\right\}$ by $b(d(A))$, and the number of triangles having one vertex in $\mathcal{A}$ by $t(\mathcal{A})$. Then using Abel-summation we infer:

$$t(\mathcal{A}) = \sum_{A \in \mathcal{A}} t(A) \leqq \sum_{1 \leqq i \leqq n} a(i) b(i) = \sum_{2 \leqq i \leqq n} A(i)\big(b(i) - b(i-1)\big). \tag{4}$$

To get (4) we used of course $b(1) = 0$. Now $b(i) = b(i-1)$ for $i \geqq 2 + \sqrt{2n}$, $b(i) - b(i-1) \leqq i-1$ for all i. Thus taking into account (3) from (4) we obtain:

$$t(\mathcal{A}) \leqq \sum_{2 \leqq i < 2+\sqrt{2n}} b_0 n(i-1)/i < b_0 n(1 + \sqrt{2n}) = O(n^{3/2}). \quad \blacksquare$$

4. The constructions

As in Remark 1 we have shown that (iii) is a consequence of Theorem 4, we have to prove only the second inequality in (ii). The construction is quite involved. Let us first resume the main idea.

For some integer m and every sequence of g integers $(t_1, \dots, t_g)$, $0 \leqq t_i < m$ for $1 \leqq i \leqq g$, we construct a different set of cardinality k, i.e. the sets will be uniquely determined by the corresponding sequence. We call t_i its i'th coordinate. The k-sets will be obtained as the union of some members of a family of pairwise disjoint a-sets and b_1-sets, which we call bricks. The bricks all carry information about the coordinates of the k-set, however they are all determined by at most $g-1$ coordinates. Thus the ground set, i.e., the union of all the bricks has cardinality $O(m^{g-1})$, while the total number of k-sets will be m^g. The information or coordinates of the bricks will be organized in such a way that if two k-sets both contain either 2 bricks of size a or one of size b_1, then they coincide in b_0 bricks of size a and one of size b_1 or entirely. Thus the possible cardinalities for pairwise intersections will be 0, a and $b_0 a + b_1 = b$, as desired.

Let f, g, b_0, b_1 as in Theorem 5 and let $I_1, \dots, I_g$ be b_0-subsets of $\{1, \dots, f\}$ satisfying $|I_j \cap I_{j'}| \in \{0, 1\}$ for $j \neq j'$.

Let m be the largest integer such that $km^{g-1} \leqq n$. Let us arrange fm^g a-element sets and gm^g b_1-element sets, all pairwise disjoint, in a rectangular array with $f+g$ columns, m^g rows. The first f columns contain the a-sets, the last g the b-sets. We index the rows by integer sequences $T=(t_1, \dots, t_g)$; $0 \leqq t_i < m$ for $1 \leqq i \leqq g$.

For every such sequence T we define a set $F(T)$ which will be the union of $f+g$ sets, one from each column. Thus $|F(T)| = k$.

For $1 \leqq i \leqq f$ the component of $F(T)$ in the i'th column will be the $T(i)$'th, where $T(i)$ is the unique sequence which has 0 in the j'th position whenever $i \in I_j$, $1 \leqq j \leqq g$, and has the same entry as T everywhere else. As $f < b_0 g$, we may assume that every i, $1 \leqq i \leqq f$, is contained in at least one I_j, $1 \leqq j \leqq g$. Thus $T(i)$ has at most $g-1$ non-zero entries.

For $f+1 \leqq i \leqq f+g$ the component of $F(T)$ in the i'th column will be the one having index $T(i)$, where $T(i)$ is the unique sequence having 0 in the $(f-i)$'th position and coinciding with T in the remaining $g-1$ positions.

We set $\mathscr{F} = \{F(T)\colon\ T=(t_1, \dots, t_g),\ 1 \leqq t_j \leqq m,\ \text{for}\ 1 \leqq j \leqq g\}$.

By definition we have used from each column at most m^{g-1} sets. Thus the union of all the $F(T)$'s has cardinality at most $(fa+gb_1)m^{g-1} = km^{g-1} \leqq n$. Of course $|\mathscr{F}| = m^g$. So we only need to show that the pairwise intersections of members of $\mathscr{F}$ have cardinalities all belonging to $\{0, a, b\}$. Let, in fact, $F(T)$ ad $F(T')$ be two members of $\mathscr{F}$. If they are disjoint, then we are done. If not, then there is at least one element of the array which they both contain. Arguing indirectly we may assume that either this is in the last g columns or they have at least 2 elements of the array in common in the first f columns. In the first case let $f+i$ be the index of this column, $1 \leqq i \leqq g$, i.e. $T(f+i) = T'(f+i)$. But T and $T(f+i)$ differ in at most one position, the i'th. Thus $T(j) = T'(j)$ whenever $j \in I_i$, and any more coincidence would imply $T = T'$. Hence $|F(T) \cap F(T')| = b_0 a + b_1 = b$, as desired.

Now the proof of Theorem 5 is complete. Now we want to describe an alternative construction, which gives $O(n^{3/2})$ sets, however it does not work in all cases covered by (ii).

We replace $m(f, b_0, \{0, 1\}) \leqq g$ by a much stronger assumption. We suppose that there exists an $(f, b_0, \{0, 1\})$-system $I_1, \ldots, I_g$ and an embedding, φ of it into a projective plane, P over the finite field of q elements, for some prime power q. Moreover, the embedding has the property that the image of I_j is contained in some line for $1 \leqq j \leqq g$, and if $r, s \in \{1, \ldots, f\}$ are both contained in at least two of the sets I_j then there exists an $I_{j'}$ which contains them both. This latter condition means that the restriction of $\{I_1, \ldots, I_g\}$ to the points of degree at least 2 is a pairwise blanced design.

By symmetry we may assume that $\varphi(1)$, $\varphi(2)$, $\varphi(3)$ are not collinear points of P. Then we can express all the other points of P in terms of these points using homogenous (barycentric) coordinates. Let $h_i(j)$ denote the i'th coordinate of $\varphi(j)$, $1 \leqq i \leqq 3$, $1 \leqq j \leqq f$. In particular, $h_i(j) = \delta_{i,j}$ for $1 \leqq j \leqq 3$.

Let $1, \ldots, f_0$ be the points of degree at least 2 in $\{1, \ldots, f\}$. For an integer t, we set $p(t) = (q^{t+1} - 1)/(q-1)$. Let m be the maximal integer satisfying $f_0 a p(m) + (f - f_0 + g) b_1 p(m)^2 \leqq n$.

Let W be the m-dimensional projective space over $GF(q)$. We form two arrays: one consisting of pairwise disjoint a-sets, and of size $p(m)$ by f_0. The second of size $p(m)^2$ by $f - f_0 + g$, which consists of pairwise disjoint sets, which are also disjoint to the sets of the first array. The sets in the first $f - f_0$ columns are of size a while all the others have size b_1. The rows are indexed in the first array by the points of W, while in the second by its pairs of points.

Now with every non-collinear triple of points $w_1, w_2, w_3 \in W$ we associate a k-set. For that in every I_j we fix two points of degree $\geqq 2$, $r(j)$, $s(j)$. For $f_0 < i \leqq f$ let $I_{j(i)}$ be the unique I_j containing i. Now the k-set $F(w_1, w_2, w_3)$ will be the union of $f + g$ sets, one from each column of each array. The index of the entry from the i'th column is $h_1(i)w_1 + h_2(i)w_2 + h_3(i)w_3$ for $1 \leqq i \leqq f_0$. For the $(f - f_0 + j)$'th column this index is the pair of indices for the columns $r(j)$, $s(j)$; while for the column t, $0 < t \leqq f - f_0$ this index is identical with that of the $(f - f_0 + j(t + f_0))$'th column.

As the number of triples w_1, w_2, w_3 is $O(n^{3/2})$ we only have to show that for different triples v_1, v_2, v_3 and w_1, w_2, w_3 the intersection of $F(v_1, v_2, v_3)$ and $F(w_1, w_2, w_3)$ has cardinality 0, a or b.

As every point of a line can be uniquely expressed as the homogenous linear combination of two given points of this line, we see that if the two sets agree in two entries of the first array, then they agree in all the other columns corresponding to the set I_j going through these two points. If they agree in a position in the second array, then they agree in all the columns corresponding to that I_j. These would give intersection of size b. However if they would agree anywhere else, then the two k-sets would be identical, as all the points of a plane can be expressed as homogenous linear combination of 3 non-collinear points. This would imply that they agree in the first 3 columns, thus $v_1 = w_1$, $v_2 = w_2$, $v_3 = w_3$, proving our claim.

Added in proof. Z. Füredi proved that there are values of a, b and k for which $m(n, k, \{0, a, b\}) = O(n^{4/3})$ holds. Most recently the author showed that for every rational p/q, $p \geqq q$, there exist k and L with $m(n, k, L) = O(n^{p/q})$.

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